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LINEAR k-BLOCKING SETS

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We point out the relationship between normal spreads and the linear k-blocking sets introduced in [9]. We give a characterisation of linear k-blocking sets proving that the projections and the embeddings of a $PG(kt,q)$ in $PG(r-1,q^t)$ are linear k-blocking sets of $PG(r-1,q^t)$. Finally, we construct some new examples.

1. Introduction

Denote by k a positive integer. A *k–blocking set* in the finite projective space $PG(r-1,s)$, $r\geq k+2$, is a set B of points such that any $(r-k-1)$-dimensional subspace contains a point of B and no k-dimensional subspace is contained in B. When $k=r-2$, we simply say that B is a *blocking set*. If for each point x of B the set $B\setminus\{x\}$ is not a k-blocking set (i.e., if there is a $(r-k-1)$-dimensional subspace of $PG(r-1,s)$ intersecting B in exactly the point x), we say that B is *minimal*. It is proved in [2] that $|B|\geq s^k+s^{k-1}+\cdots+s+1+s^{k-1}\sqrt{s}$. The smallest k-blocking sets in $PG(r-1,s)$, $s>2$, are the cones $\mathcal{B}(U,B_0)$ with vertex a $(k-2)$-dimensional subspace U of $PG(r-1,s)$, and with base a blocking set B_0 of the smallest possible size in a plane E of $PG(r-1,s)$ disjoint from U (see [6]). If $s=q^2$, then $B_0=E_0$ is a Baer-subplane of E and $\mathcal{B}(U,E_0)$ has size $s^k+s^{k-1}+\cdots+s+1+s^{k-1}\sqrt{s}$ (see [2] Lemma 4). We remark that all the abovementioned k-blocking sets are blocking sets of a $(k+1)$-dimensional subspace and they contain $(k-1)$-dimensional subspaces.

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A minimal blocking set B of $PG(2,s)$ has order $|B|=s+N\geq s+|B\cap l|$ where l is a line of the plane. If $N<\frac{s+3}{2}$, we call B a *small* minimal blocking set. If there is a line l contaning exactly N points of B, we call l a *Rédei line* and B is said to be a *Rédei minimal blocking set*. The reader can see [13] for a detailed exposition of the topic. Examples of minimal blocking sets of $PG(2,s)$ $(s=q^t)$ can be constructed in the following way. Let $f:GF(s)\mapsto GF(s)$ be a $GF(q)$-linear map. The set $B_f=R\cup D_R$, where $R=\{(a,f(a),1)\mid a\in GF(s)\}$ and $D_R=\{(a,f(a),0)\mid a\in GF(s)\}$, is a Rédei minimal blocking set of $PG(2,s)$ containing q^t+N points (see [3] or [13]).

Let $B=R\cup D_R$ be a Rédei minimal blocking set of $PG(2,s)(s=p^n$, p prime) where R contains exactly s points of the affine plane $AG(2,s)$ obtained from $PG(2,s)$ fixing a Rédei line l as line at infinity, and D_R is the set of directions determined by the set R. We can suppose that the affine point $(0,0)$ belongs to R. Let e be the largest integer such that each line with slope in D_R meets R in a number of points which is a multiple of p^e. If either $p^e>3$ or $p^e=3$ and $N=p^n/3+1$, then there is a $GF(p^e)$-linear map $f:GF(s)\mapsto GF(s)$ such that $B=B_f$ (see [1]).

A new class of k-blocking sets, called *linear*, has been introduced in [9] using normal spreads of projective finite spaces, proving that B_f is always a linear blocking set, and a class of examples of non-Rédei small minimal linear blocking sets of $PG(2,q^t)$ has been constructed in [11].

The aim of this paper is to point out the relationship between normal spreads and linear k-blocking sets. We prove that a k-blocking set of minimal size is linear when s is a square. Using the characterisation of linear k-blocking sets obtained in Section 4, we can prove that all the projections and all the embeddings of $PG(kt,q)$ in $PG(r-1,q^t)$ $(r\geq k+2,\ kt>r-1)$ are linear k-blocking sets of $PG(r-1,q^t)$. Finally, we construct some new examples.

2. Normal spreads

Let $PG(V,F)$ be the projective space defined by the lattice of the vector subspaces of the vector space V over the field F. Denote by the same symbol both a vector subspace of V and the element of $PG(V,F)$ defined by it. We say that an element T of $PG(V,F)$ has *rank* t and *dimension* $t-1$ if T has dimension t as a vector space over F. If V has finite dimension n over $F=GF(q)$, we write $PG(n-1,q)$ instead of $PG(V,F)$.

Let Σ^* be a projective space. A subset Σ of points of Σ^* is a *subgeometry* of Σ^* if there is a set $\mathcal{L}$ of subsets of Σ with the following properties:

(1) each element of $\mathcal{L}$ is contained in a line of Σ^*;

(2) $(\Sigma, \mathcal{L})$ is a projective space;
(3) if a line l of Σ^* contains two points of Σ, then $l \cap \Sigma \in \mathcal{L}$;
(4) no line of Σ^* belongs to $\mathcal{L}$.

Let $\Sigma = PG(V, GF(q)) = PG(n-1, q)$ be a subgeometry of $\Sigma^* = PG(V^*, GF(q^t)) = PG(n-1, q^t)$. We say that Σ is a *canonical* subgeometry of Σ^* when $V^* = GF(q^t) \otimes V$.

Let Σ be a canonical subgeometry of Σ^*. For each subspace S^* of Σ^* the set $S = S^* \cap \Sigma$ is a subspace of Σ whose rank is at most equal to the rank of S^*. We say that a subspace S of Σ^* is a *subspace* of Σ whenever S and S^* have the same rank. If σ is a semilinear collineation of Σ^* of order t fixing pointwise Σ, then S^* is a subspace of Σ if and only if S^* is fixed by σ.

Let V_1, V_2 be finite dimensional vector spaces over the field F, of dimension t and r respectively. The vector space $V = V_1 \otimes V_2$ has dimension rt. Let $\Sigma = PG(V, F) = PG(rt-1, F)$. A *Segre variety* of type (t, r) is the set $S_{t,r} = \{a_1 \otimes a_2 \mid a_i \in V_i \setminus \{0\} (i = 1, 2)\}$. For each non-zero vector $a_i \in V_i$ $(i = 1, 2)$ the vector subspaces $\langle a_1 \rangle \otimes V_2$ and $V_1 \otimes \langle a_2 \rangle$ define respectively a subspace of rank r and a subspace of rank t of $PG(rt-1, q)$. Let $\mathcal{R} = \{V_1 \otimes \langle a_2 \rangle \mid a_2 \in V_2; a_2 \neq 0\}$ and $\mathcal{R}^* = \{\langle a_1 \rangle \otimes V_2 \mid a_1 \in V_1; a_1 \neq 0\}$. It is easy to prove that each point of $S_{t,r}$ belongs to exactly one element of $\mathcal{R}$ and one element of $\mathcal{R}^*$. We call $\mathcal{R}$ a $(t-1)$-regulus of $PG(rt-1, F)$ and $\mathcal{R}^*$ the set of the transversal subspaces of $\mathcal{R}$. If A is an element of $\mathcal{R}^*$ and B is a k-dimensional subspace of A, then for each X of $\mathcal{R}$ the set of points $X_B = \{Y \cap X \mid Y \in \mathcal{R}, Y \cap A \in B\}$ is a k-dimensional subspace of X.

If $A_0, A_1, \cdots, A_r$ are $r+1$ subspaces $PG(rt-1, q)$ of dimension $t-1$ such that each r of them span $PG(rt-1, q)$, then there is a unique $(t-1)$-regulus containing $A_0, A_1, \cdots, A_r$.

The reader can see e.g. [5] for more details on Segre varieties.

A $(t-1)$-*spread* $(t > 1)$ of a projective space $PG(n-1, q)$ is a family $\mathcal{S}$ of mutually disjoint subspaces of rank t such that each point of $PG(n-1, q)$ belongs to an element of $\mathcal{S}$. It has been proved by Segre [12] that $(t-1)$-spreads of $PG(n-1, q)$ exist if and only if $n = rt$.

Let $r > 2$. A $(t-1)$-spread $\mathcal{S}$ is said to be *normal* if it induces a spread in any subspace generated by two elements of $\mathcal{S}$ (i.e., if $T = \langle A, B \rangle$ with A, B in $\mathcal{S}$, then an element of $\mathcal{S}$ is either disjoint from T or contained in T)[1].

Let $\Sigma = PG(rt-1, q)$ be a canonical subgeometry of $\Sigma^* = PG(rt-1, q^t)$ $(t > 1)$ and let σ be a semilinear collineation of Σ^* of order t which fixes Σ pointwise. There is a subspace $\Pi = PG(r-1, q^t)$ disjoint from Σ such that Σ^* is spanned by Π, Π^σ, $\ldots$ $\Pi^{\sigma^{t-1}}$ and, for each point x of Π, $L(x) = \langle x^{\sigma^i} \mid i = 0, 1, 2, \ldots, t-1 \rangle$ is a subspace of Σ of rank t. Then it is easy

[1] This spreads are called *geometric* in [12].

to prove that $\mathcal{S}=\{L(x)\mid x\in\Pi\}$ is a $(t-1)$-spread of $PG(rt-1,q)$ (see, e.g, [12]). If m is a line of Π, then $\mathcal{S}_m=\{L(x)\mid x\in m\}$ is a $(t-1)$-spread of the subspace $T_m=\langle m,m^\sigma,\dots,m^{\sigma^{t-1}}\rangle$ of rank $2t$. The $(t-1)$-spread $\mathcal{S}$ has the following property: if a subspace T of rank $2t$ of $PG(rt-1,q)$ contains two elements of $\mathcal{S}$, then there is a line m of Π such that $T=T_m$. Therefore, when $r>2$, $\mathcal{S}$ is a normal spread of Σ, and by [12] all normal spreads of Σ can be constructed in this way.

A subset π of $\Pi=PG(r-1,q^t)$ is a canonical subgeometry of Π if and only if $\mathcal{R}=\{L(x)\mid x\in\pi\}$ is a $(t-1)$-regulus of Σ (see [12] or [7] Lemma 25.6.8).

Let $\mathcal{L}$ be the set of all the subspaces of $PG(rt-1,q)$ of rank $2t$ joining two elements of $\mathcal{S}$. Let $P(\mathcal{S})$ be the incidence structure, whose points and lines are respectively the elements of $\mathcal{S}$ and the elements of $\mathcal{L}$, and whose incidence is the usual one of $PG(rt-1,q)$. Then $P(\mathcal{S})$ is isomorphic to $\Pi=PG(r-1,q^t)$ via the isomorphism α defined by $x\mapsto L(x)$ and $m\mapsto T_m$ as remarked by R.C. Bose (see [4] or [12]).

Next we give a different construction for normal spreads.

Let V be an r-dimensional vector space over $GF(q^t)$, and let $\Pi=PG(r-1,q^t)=PG(V,GF(q^t))$. Regarding V as a vector space of dimension rt over $GF(q)$, each point x of $PG(r-1,q^t)$ defines a $(t-1)$-dimensional subspace $P(x)$ of the projective space $PG(V,GF(q))=PG(rt-1,q)$, and each line l of $PG(r-1,q^t)$ defines a $(2t-1)$-dimensional subspace $P(l)$ of $PG(rt-1,q)$.

Let $\mathbf{S}$ be the set of all the $(t-1)$-dimensional subspaces $P(x)$ where x is a point of $PG(r-1,q^t)$. Then $\mathbf{S}$ is a $(t-1)$-spread of $PG(rt-1,q)$. Moreover, if U is a $(2t-1)$-dimensional subspace of $PG(rt-1,q)$ containing two elements of $\mathbf{S}$, then a $(t-1)$-spread is induced by $\mathbf{S}$ in U, i.e. $U=P(l)$ for some line l of $PG(r-1,q^t)$. This implies that $\mathbf{S}$ is a normal $(t-1)$-spread.

For each λ in $GF(q^t)$, let τ_λ be the collineation of $\Pi=PG(r-1,q^t)$ defined by the linear map of V to itself which maps $v\mapsto\lambda v$ for all vectors v of V. Note that τ_λ fixes all the points of Π. Also $G=\{\tau_\lambda:\lambda\in GF(q^t)\}$ defines a subgroup of $PGL(rt,q)$ of order $(q^t-1)/(q-1)$, which fixes all the elements $P(x)$ of $\mathbf{S}$ and acts sharply transitively on the points of $P(x)$. Moreover, Π is isomorphic to $P(\mathbf{S})$ via the isomorphism P defined by $x\mapsto P(x)$ and $l\mapsto P(l)$.

3. Linear *k*-blocking sets

Let $\mathcal{S}$ be a normal spread of $\Sigma=PG(rt-1,q),(t>1)$ and let $P(\mathcal{S})\simeq PG(r-1,q^t)$ be the $(r-1)$-dimensional projective space constructed using $\mathcal{S}$. We recall that a $(h-1)$-dimensional subspace X of $PG(r-1,q^t)$ is represented in $P(\mathcal{S})$ by a $(ht-1)$-subspace T_X of Σ such that $\mathcal{S}_X = \{L(x) \mid x \in X\}$ is a spread of T_X.

Theorem 1. *Let k be a positive integer such that $r \geq k+1$, and let L be a kt-dimensional subspace of Σ. Define*

$$\mathcal{B}_L = \{A \in \mathcal{S} \mid A \cap L \neq \emptyset\}$$

If L is not contained in T_Y for all k-dimensional subspaces Y of $PG(r-1,q^t)$, then $\mathcal{B}_L$ is a k-blocking set of $P(\mathcal{S}) \simeq PG(r-1,q^t)$.

Proof. If X is a $(r-k-1)$- dimensional subspace of $PG(r-1,q^t)$, then T_X intersects L, i.e. there is an element of $\mathcal{S}_X$ in $\mathcal{B}_L$. Hence all $(r-k-1)$-dimensional subspaces contain an element of $\mathcal{B}_L$.

Let Y be a k-dimensional subspace of $PG(r-1,q^t)$ and let $\mathcal{S}_Y$ be the spread of T_Y induced by $\mathcal{S}$. Then $\mathcal{S}_Y$ is contained in $\mathcal{B}_L$ if and only if either L is contained in T_Y and all elements of $\mathcal{S}_Y$ intersect L or $L \cap T_Y \neq L$ and L has at least a point in common with each element of $\mathcal{S}_Y$. If $L \cap T_Y$ is different from L, then $L \cap T_Y$ has dimension at most $kt-1$, and $|L \cap T_Y| \leq \frac{q^{kt}-1}{q-1}$. As $\mathcal{S}_Y$ has order $\frac{q^{t(k+1)}-1}{q^t-1} > \frac{q^{kt}-1}{q-1}$, not all elements of $\mathcal{S}_Y$ intersect L. ∎

In the hypotheses of Theorem 1, we call $\mathcal{B}_L$ a *linear* k-blocking set of $PG(r-1,q^t)$.

We remark that the subspace L is not uniquely defined by $\mathcal{B}_L$ because $\mathcal{B}_L = \mathcal{B}_M$ with $M = L^\tau$ for each element τ of the group G. Thus, for each element A of $\mathcal{B}_L$, the subspaces $L \cap A$ and $M \cap A$ have the same dimension for all τ in G. In particular, if $L \cap A$ is a point for some element of $\mathcal{B}_L$, then there are $\frac{q^t-1}{q-1}$ subspaces of dimension kt defining the same linear k-blocking set. If each element of $\mathcal{B}_L$ intersects L in a point, then $\mathcal{P} = \{L^\tau \mid \tau \in G\}$ is a partial spread because each point of an element of $\mathcal{B}_L$ belongs to exactly one of the subspaces L^τ.

If l is a line of L and X and Y are two elements of $\mathcal{B}_L$ incident with a point of l, then $\mathcal{R} = \{Z \in \mathcal{B}_L \mid Z \cap l \textit{ is a point}\}$ is a $(t-1)$-regulus of the spread induced by $\mathcal{S}$ on the $(2t-1)$-dimensional subspace $\langle X, Y\rangle$, whose transversals are the lines l^τ for τ in G.

Corollary 1. *For $k = r-2$, the blocking set $\mathcal{B}_L$ of $PG(r-1,q^t)$ is minimal.*

Proof. If $\mathcal{B}_L$ is not minimal, there is an element $A = L(x)$ of $\mathcal{B}_L$ such that for each line m of $\Pi = PG(r-1,q^t)$ incident with x the subspace T_m intersects an element B_m of $\mathcal{B}_L$ different from A. Let a be a fixed point of $A \cap L$. If b_m is a point of B_m, let $l_m = \langle a, b_m\rangle$ be the line of L joining a and b_m. If $m' \neq m$, then l_m and $l_{m'}$ are distinct because A is the intersection of T_m and $T_{m'}$. Thus we have $\frac{q^{t(r-1)}-1}{q^t-1}$ lines l_m. As L has dimension $(r-2)t$, the number of the lines of L incident with a point is $\frac{q^{t(r-2)}-1}{q-1} < \frac{q^{t(r-1)}-1}{q^t-1}$. Hence we have a contradiction. ∎

Corollary 2 ([9] Theorem 10). *For each $GF(q)$-linear function f from $GF(q^t)$ to itself, the Rédei blocking set B_f of $PG(2,q^t)$ is linear.* ∎

It has been proved in [11] that there are linear blocking sets of $PG(2,q^t)$, $t>3$, which are not of Rédei type.

Corollary 3. *Denote by $\mathcal{B}(U,B_0)$ the cone of $PG(r-1,q^t)$ with vertex the $(k-2)$-dimensional subspace U of $PG(r-1,q^t)$, $t>1$, and base a blocking set B_0 in a plane E of $PG(r-1,q^t)$ disjoint from U. If B_0 is a linear blocking set of the plane E then $\mathcal{B}(U,B_0)$ is a linear k-blocking set of $PG(r-1,q^t)$.*

Proof. Let $PG(r-1,q^t) = PG(V,GF(q^t))$ and let $\mathbf{S}$ be the $GF(q)$-linear representation of $PG(r-1,q^t)$ in $PG(V,GF(q)) = PG(tr-1,q)$. Then U and E define respectivelly a $(tk-t-1)$-dimensional subspace T_U and a $(3t-1)$-dimensional subspace T_E of $PG(tr-1,q)$.

If B_0 is a linear blocking set of E, then there is a t-dimensional subspace A of T_E such that $B_0 = \{x \mid L(x) \cap A \neq \emptyset\}$. Let L be the subspace of $PG(tr-1,q)$ joining T_U and A. As U and E are disjoint, the subspaces T_U and A are skew. Then L has dimension tk and $\mathcal{B}(U,B_0) = \mathcal{B}_L$. ∎

Let E_0 be a Baer subplane of $E = PG(2,q^2)$. As E_0 is a Rédei blocking set of $E = PG(2,q^2)$, E_0 is a linear blocking set of E by [1] and Corollary 2. Hence, $\mathcal{B}(U,E_0)$ is a linear k-blocking set of $PG(r-1,q^2)$.

4. Characterisation of linear k-blocking sets

Let $\mathcal{S}$ be a normal spread of $\Sigma = PG(rt-1,q)$. Suppose that $\Sigma = PG(rt-1,q)$ is a canonical subgeometry of $\Sigma^* = PG(rt-1,q^t)$, and σ is the semilinear collineation of Σ^* which fixes Σ pointwise. Let $\Pi = PG(r-1,q^t)$ be a subspace disjoint from Σ such that Σ^* is spanned by Π, Π^σ, $\ldots \Pi^{\sigma^{t-1}}$ and $\mathcal{S} = \{L(x) \mid x \in \Pi\}$. If $\Pi = PG(V,GF(q^t))$, then we can suppose $\Sigma^* = \{(x_1, x_2^\sigma, \ldots, x_t^{\sigma^{t-1}}) \mid x_1, x_2, \ldots, x_t \in V\}$, and $\Sigma = \{(x, x^\sigma, \ldots, x^{\sigma^{t-1}}) \mid x \in V\}$.

Suppose $r \geq k+2$. Let L be a kt-dimensional subspace of Σ which is not contained in T_Y for all k-dimensional subspaces Y of $\Pi = PG(r-1,q^t)$, i.e. $\mathcal{B}_L$ is a linear k-blocking set. Denote by B_L the set of all the points x of Π

such that $L(x)$ belongs to $\mathcal{B}_L$. By the isomorphism α between Π and $P(\mathcal{S})$ defined in §2, B_L is a k-blocking set of Π, which is also called *linear*. Note that $\alpha(B_L)=\mathcal{B}_L$.

Theorem 2. *Let $r\geq k+2$. There is a kt-dimensional subspace L of Σ which is not contained in T_Y for all k-dimensional subspaces Y of $\Pi=PG(r-1,q^t)$ if and only if there is a subset W of V, which is a $(kt+1)$-dimensional vector space over $GF(q)$, such that:*

(a) a point of Π belongs to B_L if and only if it is defined by a vector of W,

(b) W is not contained in any $GF(q^t)$-vector subspace of V of dimension $k+1$ over $GF(q^t)$.

Proof. Let $W=\{x\in V\,|\,(x,x^\sigma,\dots,x^{\sigma^{t-1}})\in L\}$. If x and y belong to W and λ is in $GF(q)$ then both $x+y$ and λx belong to W. This is equivalent to saying that W is a $(kt+1)$-dimensional vector space over $GF(q)$. Therefore L is a kt-dimensional subspace of Σ if and only if there is a $GF(q)$-vector subspace W of V of dimension $kt+1$ such that $L=\{(x,x^\sigma,\dots,x^{\sigma^{t-1}})\,|\,x\in W\}$.

By definition a point y of Π belongs to B_L if and only if $L(y)$ belongs to $\mathcal{B}_L$ if and only if $(x,x^\sigma,\dots,x^{\sigma^{t-1}})\in L$ with $x=\mu y$ for some $\mu\in GF(q^t)$ and $x\in W$ if and only if the point y of Π is defined by a vector of W.

Moreover W is contained in a $GF(q^t)$-vector space of V of dimension $k+1$ over $GF(q^t)$ if and only if B_L is contained in a k-dimensional subspace Y of Π if and only if L is contained in T_Y. ∎

In the hypothesis of Theorem 2, a point of B_L can be defined by different vectors of W. In particular if $\Pi=PG(2,q^t)$ is a plane, then a linear blocking set of Π is defined by a $(t+1)$-dimensional $GF(q)$-vector space of V.

Corollary 4. *A canonical subgeometry of $PG(kt,q^t)$ is a linear k-blocking set. If $r=kt+1$, then a linear k-blocking set B of $PG(r-1,q^t)$ is a canonical subgeometry if and only if $\langle B\rangle=PG(r-1,q^t)$.*

Proof. If $\Sigma=PG(W,GF(q))$ is a canonical subgeometry of $PG(kt,q^t)=PG(V,GF(q^t))$ then Σ is a linear k-blocking set by Theorem 2.

If B is a linear k-blocking set of $PG(kt,q^t)$, let W be the $GF(q)$- vector space of dimension $kt+1$ associated with B. As W has dimension $kt+1$ and $\langle B\rangle=PG(kt,q^t)$ a basis of W is also a basis of V, i.e. B is a canonical subgeometry of $PG(kt,q^t)$. ∎

5. Projections and embeddings

Let $\Sigma = PG(m,q)$ be a canonical subgeometry of $\Sigma^* = PG(m,q^t)$. Suppose there is a $(m-r)$-dimensional subspace Λ^* of Σ^* disjoint from Σ. Let Λ be an $(r-1)$-dimensional subspace of Σ^* disjoint from Λ^*, and let $\Gamma = \{x \textit{ is a point of } \Lambda \mid \exists y \in \Sigma : x = \langle \Lambda^*, y\rangle \cap \Lambda\}$ be the *projection* of Σ from Λ^* to $\Lambda = PG(r-1,q^t)$. If each line of Σ is disjoint from Λ^*, we call Γ an *embedding* of $PG(m,q)$ in Λ. Let $p_{\Lambda^*,\Lambda,\Sigma}$ be the map from Σ on Γ defined by $x \mapsto \langle \Lambda^*, x\rangle \cap \Lambda$ for each point x of Σ.

Lemma 1. *The map $p_{\Lambda^*,\Lambda,\Sigma}$ is a bijection if and only if Γ is an embedding of $\Sigma = PG(m,q)$ in Λ. No proper subspace of Λ contains Γ.*

Proof. By defintion $p_{\Lambda^*,\Lambda,\Sigma}$ is surjective. If x and y are distinct points of Σ, then $p_{\Lambda^*,\Lambda,\Sigma}(x) = p_{\Lambda^*,\Lambda,\Sigma}(y) = z$ if and only if the subspace $\langle \Lambda^*, z\rangle$ contains x and y. This is equivalent to say that the line joining x and y intersects Λ^*.

If Γ is contained in a hyperplane H of Λ, then Σ is contained in the hyperplane $\langle \Lambda^*, H\rangle$ of Σ^*. As Σ is a canonical subgeometry of Σ^*, this is impossible. ∎

Let $\Sigma^* = PG(V^*, GF(q^t))$ and $\Sigma = PG(V, GF(q))$ with $V^* = GF(q^t) \otimes V$. Denote by X and Y the vector subspaces of V^* which define respectively Λ^* and Λ. Note that $dim_{GF(q)} X \oplus V = (m-r+1)t + m + 1$ and $dim_{GF(q)} Y = rt$. Therefore, $W = Y \cap (X \oplus V)$ is a $GF(q)$-subspace of dimension $m+1$ of Y, and the points of Γ are defined by the vectors of W.

Theorem 3. *If Γ is a projection of $PG(m,q)$ in $\Lambda = PG(r-1,q^t)$ $(t>1)$ and $m = kt$, with $r \geq k+2, k>0$, then Γ is a linear k-blocking set of Λ. When Γ is an embedding of $PG(m,q)$ in $\Lambda = PG(r-1,q^t)$, Γ has size $q^{kt}+q^{kt-1}+\dots+q+1$ and does not contain any line of Λ.*

Proof. Any $(r-k-1)$-dimensional subspace of Λ contains a point of Γ because it is defined by a $GF(q^t)$-vector subspace of Y of dimension $r-k$ over $GF(q^t)$ and W has dimension $kt+1$ over $GF(q)$.

Suppose that a k-subspace M of Λ is contained in Γ. As $\langle \Lambda^*, M\rangle \cap \Sigma$ is a subspace of Σ, it contains $\frac{q^{h+1}-1}{q-1}$ points, where h is the dimension of $\langle \Lambda^*, m\rangle \cap \Sigma$. The number of points of $\langle \Lambda^*, M\rangle \cap \Sigma$ is greater than or equal to the number of point of its projection $\langle \Lambda^*, M\rangle \cap \Gamma = M$; i.e., $\frac{q^{h+1}-1}{q-1} \geq \frac{(q^t)^{k+1}-1}{q^t-1}$. This implies $h > tk$ because $t>1$. As the dimension of Σ is kt and h is the dimension of a subspace of Σ, we have a contradiction.

Suppose Γ be an embedding. As no line of Σ intersects Λ^*, Γ contains $q^{kt}+q^{kt-1}+\dots+q+1$ points.

Suppose that a line m of Λ is contained in Γ. Then $\langle \Lambda^*, m\rangle$ intersects Σ in exactly q^t+1 points because the map $p_{\Lambda^*,\Lambda,\Sigma}$ is a bijection. As $\langle \Lambda^*, m\rangle \cap \Sigma$ is a subspace of Σ, it contains $\frac{q^{h+1}-1}{q-1}$ points where h is the dimension of $\langle \Lambda^*, m\rangle \cap \Sigma$. As the map $p_{\Lambda^*,\Lambda,\Sigma}$ from Σ into Γ is a bijection, it must be $\frac{q^{h+1}-1}{q-1} = q^t+1$. As this is impossible, we have a contradiction. Therefore no line of Λ is contained in Γ. ∎

We remark that if all the planes of Σ are disjoint from Λ^*, then Γ is a subgeometry of Λ and, by [8], all subgeometries of Λ isomorphic to $PG(m,q)$ can be constructed in this way. In [10], we have proved that any linear k-blocking set not contained in a hyperplane is some projection.

We conclude this section with an example of 1-blocking set defined by an embedding.

Let $\Sigma^* = PG(t,q^t)$, $t>2$, and let $(x_0,x_1,x_2,\cdots,x_t)$ be the homogeneous coordinates of a point of Σ^*. If σ is the collineation of Σ^* defined by $\sigma : (x_0,x_1,x_2,\cdots,x_t) \mapsto (x_0^q, x_t^q, x_1^q, \ldots, x_{t-1}^q)$, then $\Sigma = \{(\alpha, x, x^q, \ldots, x^{q^{t-1}}) \mid \alpha \in GF(q), x \in GF(q^t)\}$ is a canonical subgeometry of Σ^* fixed pointwise by σ. The point $(0,1,0,\ldots,0)$ of Σ^* cannot be contained in a subspace U of Σ of dimension $h < t-1$ because $U^\sigma = U$ implies $(0,1,0,\ldots,0)^{\sigma^i} \in U$ for $i=0,1,\ldots,t-1$ (i.e. U is the hyperplane with equation $x_0 = 0$). If Λ is the hyperplane of Σ^* of equation $x_1 = 0$, the projection of Σ from the point $(0,1,0,\ldots,0)$ on Λ is

$$\Gamma = \{(\alpha, 0, x^q, \ldots, x^{q^{t-1}}) \mid \alpha \in GF(q), x \in GF(q^t)\}.$$

Then Γ is a 1-blocking set of $\Lambda = PG(t-1,q^t)$ by Theorem 2.

For $t=3$ the line $x_0 = x_1 = 0$ of $\Lambda = PG(2,q^3)$ is a Rédei line of Γ containing q^2+q+1 points of Γ.

If $t>3$, then Γ is a subgeometry of $\Lambda = PG(t-1,q^t)$ and a line of Λ contains 0, 1 or $q+1$ points of Γ.

6. An example of k-blocking set

In this section we always suppose t and k are two positive integers such that $t \geq 3$ and $r = k(t-1) \geq 3$.

Let $\Sigma = PG(kt-1,q)$ be a canonical subgeometry of $\Sigma^* = PG(kt-1,q^t)$, and let σ be the semilinear collineation of Σ^* which fixes Σ pointwise. Let $\Pi = PG(k-1,q^t)$ be a subspace disjoint from Σ such that Σ^* is spanned by Π, Π^σ, ... $\Pi^{\sigma^{t-1}}$ and such that for each point x of Π, $L(x) = \langle x^{\sigma^i} \mid i = 0,1,2,\ldots,t-1\rangle$ is a subspace of Σ of rank t.

Then no line of Σ intersects $\Pi=\Lambda^*$. Let $\Lambda=\langle \Pi^{\sigma^i} \mid i=1,2,\dots,t-1\rangle = PG(r-1,q^t)$ and let Γ be the projection of Σ from Λ^* into Λ. As Γ is an embedding of $\Sigma=PG(kt-1,q)$, it contains $\frac{q^{kt}-1}{q-1}$ points and it is not contained in a hyperplane of Λ.

Let $\mathbf{S}$ be the $GF(q)$-linear representation of $\Lambda=PG(Y,GF(q^t))=PG(r-1,q^t)$ in $PG(Y,GF(q))=PG(rt-1,q)$. As the embedding Γ is defined by a $GF(q)$-vector subspace of Y of dimension kt over $GF(q)$, it defines a $(kt-1)$-dimensional subspace M of $PG(rt-1,q)$ such that for each point x of Γ the element $P(x)$ of $\mathbf{S}$ intersects M in exactly a point.

Theorem 4. *Let u be a fixed point of Γ, $y=P(u)\cap M$ and let z be a fixed point of $P(u)$ different from y. If L is the kt-dimensional subspace joining M and z, then $\mathcal{B}_L$ is a k-blocking set of $P(\mathbf{S})=PG(r-1,q^t)$ which is not contained in a hyperplane.*

If $t>3$, then any line of $P(\mathbf{S})$ contains at most q^2+q+1 elements of $\mathcal{B}_L$, i.e. $\mathcal{B}_L$ does not contain any line of $P(\mathbf{S})$. Moreover, $\mathcal{B}_L$ has order $q^{kt}+q^{kt-1}+\cdots+q^2+1$.

If $k=1$ and $t=4$ then $\mathcal{B}_L$ is a non-Rédei blocking set of the plane $PG(2,q^4)$ contaning a subgeometry isomorphic to $PG(3,q)$.

Proof. If there is a hyperplane H of Λ such that $B_L=\{x \mid P(x)\in\mathcal{B}_L\}\subset H$, then Γ is contained in H. As this is impossible, $\mathcal{B}_L$ is not contained in the hyperplane $\mathbf{S}_H=\{P(x) \mid x\in H\}$ of $P(\mathbf{S})$.

If B_L contains a k-subspace U of Λ, then Γ is contained in U. As Γ is an embedding, this is impossible by Lemma 1. By Theorem 1 we have proved that $\mathcal{B}_L$ is a k-blocking set.

Let $t>3$. In this case Γ is a subgeometry of Λ because no plane of Σ intersects Π. Therefore a line m of Λ contains at most $q+1$ point of Γ. If T_m is the $GF(q)$-linear representation of m, then $T_m\cap M$ is at most a line because each element $P(x)$ of $\mathbf{S}$ with $x\in\Gamma$ have exactly a point in common with M. As M is a hyperplane of L, we have that $L\cap T_m$ is at most a plane. Moreover, $|\mathbf{S}_m\cap\mathcal{B}_L|\le q^2+q+1$. As $|\mathbf{S}_m|=q^t+1>q^2+q+1$, no line is contained in $\mathcal{B}_L$.

Let N be a fixed hyperplane of M not incident with the point $y\in P(u)$. If $x\in P(a)$ $(a\neq u)$ is a point of N then the plane $\langle x,y,z\rangle$ of L is contained in the subspace T_m where m is the line of Λ joining the points u and a. As $P(u)$ intersects $\langle x,y,z\rangle$ in the line $\langle y,z\rangle$ each point of $\langle x,y,z\rangle$ not in $\langle y,z\rangle$ belongs to exactly one element of $\mathbf{S}_m$, i.e. there are exactly q^2+1 elements of $\mathbf{S}_m$ contaning a point of $\langle x,y,z\rangle$. As Γ is a subgeometry, $T_m\cap M$ is at most a line and $T_m\cap L=\langle x,y,z\rangle$, i.e. T_m contains exactly q^2+1 elements of $\mathcal{B}_L$.

If w is a point of N different from x belonging to $P(b)$, the line $n=\langle u,b\rangle$ of Λ intersects m in u. Therefore $P(u)$ is the unique element of $\mathcal{B}_L$ incident with a point of $\langle x,y,z\rangle$ and with a point of $\langle w,y,z\rangle$.

As we have $q^{kt-2}+q^{kt-3}+\cdots+q+1$ planes of type $\langle x,y,z\rangle$ where x is a point of N, the order of $\mathcal{B}_L$ is $q^{kt}+q^{kt-1}+\cdots q^3+q^2+1$.

If $k=1$ and $t=4$, then $r=3$ and $\mathcal{B}_L$ is a blocking set of the plane $P(\mathbf{S})=PG(2,q^4)$ of order $q^4+q^3+q^2+1$ such that no line of $P(\mathbf{S})$ contains q^3+q^2+1 points of $\mathcal{B}_L$. ∎

Note that the non-Rédei blocking set of $PG(2,q^4)$ constructed in Theorem 4 is one of the examples construted in [11]. Also, for $k=2$ and $t=3$, we have constructed a blocking set of $PG(3,q^3)$.

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